Analysis 2, Summer 2024

List 7

Slope fields, autonomous ODEs, separable ODEs

168. Classify each equation below (the ODE is not shown) as an “implicit solution” or an “explicit solution”.

   (a) $x^2 = \sin(3t)$ $\boxed{\text{implicit}}$

   (b) $y = e^t + C$ $\boxed{\text{explicit}}$ (assuming the ODE was for $y(t)$)

   (c) $y = xe^x - 5y^3$ $\boxed{\text{implicit}}$ because y appears on both sides of the $=$ sign

   (d) $y = xe^x - 5x^3$ $\boxed{\text{explicit}}$ (assuming the ODE was for $y(x)$)

   (e) $\frac{-1}{x^5} = t^7$ $\boxed{\text{implicit}}$

   (f) $\ln(y) = 9x$ $\boxed{\text{implicit}}$

169. Match the following ODEs to their slope fields.

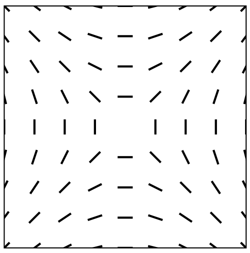
   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x}{y}$ $\boxed{\text{I}}$

   (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{x}$ $\boxed{\text{II}}$

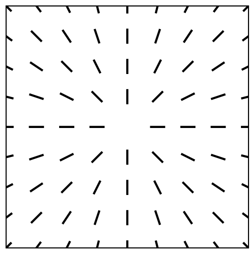
   (c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-x}{y}$ $\boxed{\text{IV}}$

   (d) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-y}{x}$ $\boxed{\text{III}}$

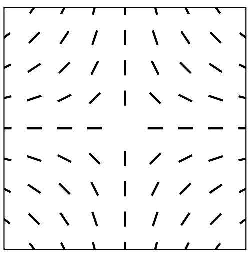
(I)



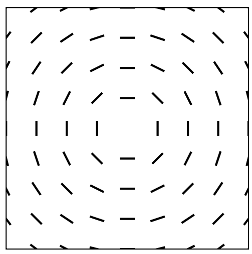
(II)



(III)



(IV)



An ordinary differential equation (ODE) for $y(x)$ is **direct** (or **directly integrable**) if it can be written in the form
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$
for some function f. An **autonomous** ODE for $y(x)$ can be written in the form
$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(y)$$
for some function g. A **separable** ODE for $y(x)$ can be written in the form
$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(y) \cdot h(x)$$
for some functions g and h.

170. Classify each ODE as “direct” or “autonomous”, and then solve it.

   (a) $y' = x^2$ $\boxed{\text{direct}}$ From $y = \int x^2 \, \mathrm{d}x$ we get $\boxed{y = \frac{1}{3}x^3 + C}$

(b) $y' = y^2$ $\boxed{\text{autonomous}}$ If y' means $\frac{dy}{dx}$, then $\frac{dy}{y^2} = dx$ leads to the solution $\boxed{y = \frac{-1}{x+C}}$. If y' means $\frac{dy}{dt}$, then $\boxed{y = \frac{-1}{t+C}}$.

(c) $y' = t^2$ Similar to (a), $\boxed{\text{direct}}$ and $\boxed{y = \frac{1}{2}t^3 + C}$

(d) $x' = x^2$ Similar to (b), $\boxed{\text{autonomous}}$ and $\boxed{x = \frac{-1}{t+C}}$

For part (b), you can assume $y = y(t)$ or you can assume $y = y(x)$. It is not clear from the ODE what the input variable is.

171. "Every directly integrable ODE is separable." Either use formulas to explain why this is true *or* give an example that shows this claim is false.

$\boxed{\text{True}}$ because we can set $g(y) = 1$ in $\frac{dy}{dx} = g(y)h(x)$.

172. "Every separable ODE is autonomous." Either use formulas to explain why this is true *or* give an example that shows this claim is false.

$\boxed{\text{False}}$. $\frac{dy}{dx} = x \cdot y$ is one example (there are many) that is separable but not autonomous.

173. Solve the autonomous ODE $x' = e^x$. $\boxed{x = -\ln(C - t)}$

☆174. Solve the autonomous ODE $y' = \sin(y)$. $\boxed{y = 2\tan^{-1}(Ce^x)}$ This is actually a very easy example *if* you know that $\displaystyle\int \frac{dy}{\sin(y)} = \ln(\tan(y)) + C$. The task is starred only because that integral is not well-known.

175. Solve the autonomous ODE $y' = k\,y^2$. Your answer should be an explicit general formula for y, but it will also use the letter k. $\boxed{y = \frac{1}{-kx+C}}$

176. (a) Solve the autonomous ODE $y' = \frac{1}{y}$. $\boxed{y = \pm\sqrt{2x+C}}$

(b) Solve the autonomous IVP $y' = \frac{1}{y}$, $y(1) = 3$. $\boxed{y = \sqrt{2x+7}}$

(c) Solve the autonomous IVP $y' = \frac{1}{y}$, $y(0) = -1$. $\boxed{y = -\sqrt{2x+1}}$

☆(d) Solve the autonomous IVP $y' = \frac{1}{y}$, $y(-1) = 0$.

$\boxed{\text{Both } y = \sqrt{2x+2} \text{ and } y = -\sqrt{2x+2}}$ are solutions. This is an unusual IVP because the initial condition uses $x = -1$ but the derivative is undefined at $x = -1$ (because $y'(-1) = \frac{1}{y(-1)} = \frac{1}{0}$).

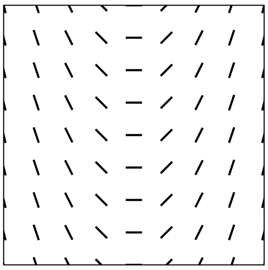
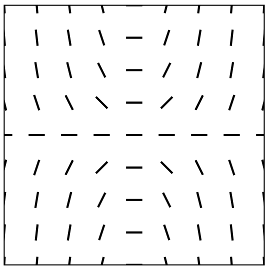
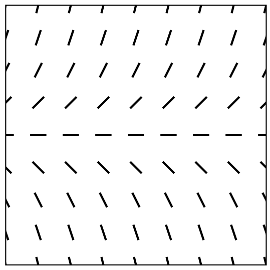
177. Solve the separable ODE $y' = 2^{y+t}$. This is an ODE, so you should give the general explicit solution, if possible.

$2^{-y}\,dy = 2^t\,dt \quad \longrightarrow \quad \frac{-1}{\ln 2}2^{-y} = \frac{1}{\ln 2}2^t + C \quad \longrightarrow \quad \boxed{y = -\log_2(C - 2^t)}$

178. Solve the separable IVP $(1 + x^3) \cdot y' = x^2\, y^2, \quad y(0) = -1$. This is an IVP, so you should give the particular explicit solution, if possible.

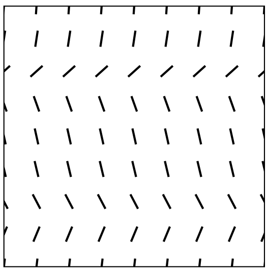
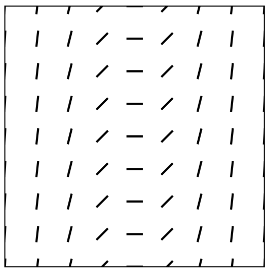
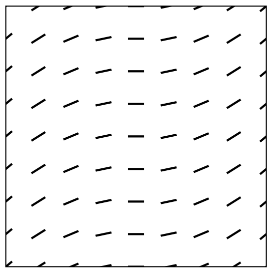
$$\boxed{y = \frac{-3}{\log\left(x^3 + 1\right) + 3}}$$

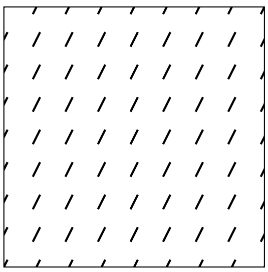
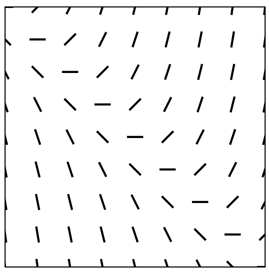
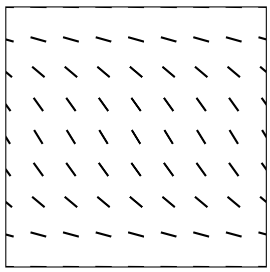
179. One of the three slope fields below corresponds to a directly integrable ODE. Which one? (A)

(A)  (B)  (C) 

180. One of the three slope fields from Task 179 corresponds to an autonomous ODE. Which one? (C)

181. For each of the slope fields below, state whether the associated ODE is directly integrable, autonomous, both, or neither.

(a)  (b)  (c) 

(d)  (e)  (f) 

(a) autonomous, (b) direct, (c) direct, (d) both, (e) neither, (f) autonomous

182. *Newton's Law of Cooling* states that the rate at which an object cools is proportional to the difference in temperature between the object and its environment. As an equation,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(T_{\text{env}} - y)$$

where k and T_{env} are constants. Find the general solution to this ODE (the answer $y(t) = \cdots$ will have t, k, T_{env}, and a new constant C in the formula).

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= -k(y - T_{\text{env}})\\
\frac{\mathrm{d}y}{y - T_{\text{env}}} &= -k\,\mathrm{d}t\\
\int \frac{\mathrm{d}y}{y - T_{\text{env}}} &= -\int k\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
\ln(y - T_{\text{env}}) &= -kt + C_1\\
y - T_{\text{env}} &= e^{-kt+C_1} = Ce^{kt}\\
&\boxed{y = T_{\text{env}} + Ce^{-kt}}
\end{aligned}$$

You can also say $y = T_{\text{env}} + C^{kt}$. The two formulas come from the fact that $\displaystyle\int \frac{\mathrm{d}y}{y - T_{\text{env}}}$ is actually $\ln\big|y - T_{\text{env}}\big|$ with absolute value bars.

183. A hot drink is cooling down according to Newton's Law of Cooling. With the external temperature at a constant 10° C, the drink has cooled from 90° to 85° in 4 minutes. How long will it take to cool down to 60°?

$y' = k(10 - y)$ leads to $y = 10 + Ce^{-kt}$. From $y(0) = 90$ we get $10 + C = 90$, so $C = 80$. From $y(4) = 85$ we get

$$\begin{aligned}
10 + 80e^{-4k} &= 85\\
80e^{-4k} &= 75\\
e^{-4k} &= 0.9375\\
-4k &= \ln(0.9375) = -0.0647\\
k &= 0.01617
\end{aligned}$$

Thus

$$y(t) = 10 + 80e^{-0.01617t}.$$

Finally, $y(t) = 10 + 80e^{-0.01617t} = 60$ occurs when $t = 29.05736$, that is, about $\boxed{\text{29 minutes total}}$, or 25 minutes after it reached 85°.

184. A boat is moving upstream, so the water applies a force $F(t) = -m\,x''(t)$ that is proportional to the velocity $v(t) = x'(t)$ of the boat. In formulas,

$$x'' = -k\,x' \qquad \text{or} \qquad v' = -k\,v.$$

The boat started its motion with velocity 1.5 m/s, and after 4 seconds it had velocity 1.00548 m/s.

From $v' = -kv$, we get $v = C^{-kt}$. From $1.5 = Ce^0$ and $1.00548 = Ce^{-4k}$ we get $C = 1.5$ and $k = 0.1$, so

$$v = x' = 1.5e^{-0.1t}.$$

Therefore $x = -15e^{-0.1t} + c$ for some c, and in order to get $x(0) = 0$ we can use $c = 15$. Thus

$$x = -15e^{-0.1t} + 15 = 15(1 - e^{-0.1t}).$$

With this formula for $x = x(t)$, we can compute...

(a) What distance had the boat traveled after 4 seconds?
... $x(4) = 15(1 - e^{-2/5}) \approx \boxed{4.95 \text{ m}}$

(b) What is the total distance the boat can go? $\lim_{t\to\infty} x(t) = \boxed{15 \text{ m}}$.

185. A cylindrical tank has a hole in the bottom, where the liquid flows out with speed proportional to the square root of the remaining volume of liquid in the tank (that is, $V' = k\sqrt{V}$). At the start the tank was full, and after 5 minutes it is half empty. How long will it take until the tank becomes completely empty?

$V' = k\sqrt{V}$ leads to $V = (\frac{1}{2}kt + c)^2$. We can assume $V(0) = 1$, which gives $V = (\frac{1}{2}kt + 1)^2$. (We could use $(\frac{1}{2}kt - 1)^2$ instead; then k would be $\frac{2-\sqrt{2}}{5}$ later, but the final answer would not change.)

The condition $V(5) = \frac{1}{2}V(0)$ means $(\frac{5}{2}k + 1)^2 = \frac{1}{2}$, so $k = \frac{-2\pm\sqrt{2}}{5}$. Although both $V = (\frac{-2+\sqrt{2}}{10}t + 1)^2$ and $V = (\frac{-2-\sqrt{2}}{10}t + 1)^2$ fit all the equations, only

$$V = \left(\frac{-2+\sqrt{2}}{10}t + 1\right)^2 = \left(\frac{2-\sqrt{2}}{10}t - 1\right)^2$$

makes physical sense. With this, $V(t) = 0$ when $\boxed{t = \frac{10}{2-\sqrt{2}} \approx 17 \text{ min } 4 \text{ sec}}$.

186. For each ODE below: if it is separable, solve it; if it is not separable, write "not separable".
Every one is separable!

(a) $y' = \sin(t)\sqrt{y}$ The general implicit solution is $2\sqrt{y} = -\cos(t) + C$, and the general explicit solution is $\boxed{y = \frac{1}{4}(C - \sin(t))^2}$.

(b) $y' = 3t^4y^5$ $\boxed{y = \pm\left(\frac{-5}{12t^5 + C}\right)^{1/4}}$

(c) $y' = 4e^{3y}\cos(t)$ $\boxed{y = \frac{-1}{3}\ln(-12\sin(t) + C)}$

(d) $y' = \dfrac{y^2+1}{yt}$ $\boxed{y = \pm\sqrt{Ct^2 - 1}}$

(e) $yty' = 1+y^2$ This is the same as part (d)!

(f) $y' = 3\ln(y)t^2y$ $\boxed{y = e^{(e^{(t^3+C)})}}$ or $y = e^{(Ce^{t^3})}$ or $y = C^{(e^{t^3})}$

(g) $\cos(y)y' = \sqrt{t\sin(y)}$ $\boxed{y = \sin^{-1}\left((\frac{4}{3}t^{3/2} + C)^2\right)}$

(h) $\dfrac{y'}{t^3} = \sin(3t)y^2$ This is difficult to solve only because $\int t^3\sin(3t)\,dt$ requires integration by parts *three times*. From

$$\int t^3\sin(3t)\,dt = \left(\frac{2t}{9} - \frac{t^3}{3}\right)\cos(3t) + \left(\frac{t^2}{3} - \frac{2}{27}\right)\sin(3t)$$

we get $\boxed{y = \dfrac{27}{(9t^3 - 6t)\cos(3t) + (9t^2 - 2)\sin(3t)}}$.

(i) $yy' = \sqrt{\sin(t)}\cos(t)e^{-y^2}$ $\boxed{y = \pm\sqrt{\ln\left(\frac{4}{3}(\sin t)^{3/2} + C\right)}}$

☆(j) $y' = (y^2 - y - 1)t$ The integral $\displaystyle\int \frac{\mathrm{d}y}{y^2 - y - 1}$ requires partial fractions, and the zeroes of the denominator are irrational: $y^2 - y - 1 = (y - \frac{1+\sqrt{5}}{2})(y - \frac{1-\sqrt{5}}{2})$. This leads to

$$\int \frac{\mathrm{d}y}{y^2 - y - 1} = \int \left(\frac{1/\sqrt{5}}{y - \frac{1+\sqrt{5}}{2}} + \frac{-1/\sqrt{5}}{y - \frac{1-\sqrt{5}}{2}} \right) \mathrm{d}y$$
$$= \frac{1}{\sqrt{5}} \ln(y - \tfrac{1+\sqrt{5}}{2}) - \frac{1}{\sqrt{5}} \ln(y - \tfrac{1-\sqrt{5}}{2}) + C$$

and therefore $\ln\left(\dfrac{y - \frac{1+\sqrt{5}}{2}}{y - \frac{1-\sqrt{5}}{2}}\right) = \frac{\sqrt{5}}{2}t^2 + C$ and $\boxed{y = \dfrac{1-\sqrt{5}}{2} - \dfrac{\sqrt{5}}{Ce^{(\sqrt{5}/2)t^2} - 1}}$.

☆(k) $y' = \dfrac{\sin(t)}{\sin(y)}$ $\cos(y) = \cos(t) + C$ is the general implicit solution. Because of issues with domain and range, the general explicit solution is onlu $\boxed{y = \cos^{-1}(\cos(t) + C)}$ (although that would earn full credit in my class). Technically all four formulas

$$\begin{aligned}
y &= 2k\pi + \cos^{-1}(\cos(t) + C), \\
y &= 2k\pi - \cos^{-1}(\cos(t) + C), \\
y &= (2k{+}1)\pi + \cos^{-1}(-\cos(t) + C), \\
y &= (2k{+}1)\pi - \cos^{-1}(-\cos(t) + C)
\end{aligned}$$

where k can be any integer, are necessary to describe *all* possible solutions.